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Critical Exponents of a Doubly Degenerate Parabolic Equation with Nonlinear Boundary*

JIANG Zhao-xin[1], KONG Ling-hua[2]

(1- School of Mathematical Sciences, Dalian University of Technology, Dalian 116024;
2- School of Science, Dalian Ocean University, Dalian 116023)

Abstract: This paper studies the critical exponents of a doubly degenerate parabolic equation with nonlinear boundary flux. The equation can be used to describe the nonstationary flow in a porous medium of fluids with a power dependence of the tangential stress on the velocity of the displacement under polytropic conditions. This study is thus meaningful in various branches of applied mathematics. We obtained the global existence exponent p_0 and critical Fujita exponent p_c by constructing various self-similar supersolutions and subsolutions of the objective doubly degenerate parabolie equation. The main results of this paper are as follows the equation exists global solutions if $0 < p \leq p_0$; the equation does not exist nontrivial and nonnegative global solutions if $p_0 < p < p_c$; the equation exists global solutions for small initial valus while does not exists global solutions otherwise if $p > p_c$.

Keywords: critical exponent; global existence; doubly degenerate parabolic equation; blow-up; nonlinear boundary flux

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1 Introduction

In the paper, we consider the critical exponents to the doubly degenerate parabolic equation

$$u_t = \left(|u_x|^{\beta}(u^m)_x\right)_x, \quad (x,t) \in \mathbb{R}^+ \times (0,T), \tag{1}$$

with

$$\begin{aligned} -|u_x|^{\beta}(u^m)_x(0,t) &= u^p(0,t), \quad && t \in [0,T), \\ u(x,0) &= u_0(x), && x \in \mathbb{R}^+, \end{aligned} \tag{2}$$

where the parameters $m \geq 1$, p, $\beta > 0$.

The equation (1) can be used to describe the nonstationary flow in a porous medium of fluids with a power dependence of the tangential stress on the velocity of the displacement under polytropic conditions, appearing in several branches of applied mathematics.

It is well known that the critical exponent problem was first studied by Fujita[1] in 1966 for Cauchy problem of the semilinear equation $u_t = \Delta u + u^p$. The critical exponent of the doubly

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degenerate equation

$$u_t = \mathrm{div}(|\nabla u|^\beta \nabla u^m) + u^p, \quad (x,t) \in \mathbb{S} = \mathbb{R}^n \times (0,\infty), \tag{3}$$

with $\beta > 0,\ m \geq 1,\ p > 0$ was proved in [2,3] that $p_c = m + \beta + \frac{\beta+2}{n}$. In [4] it was determined that the degenerate problem

$$u_t = \left(|u_x|^\beta u_x\right)_x, \quad (x,t) \in \mathbb{R}^+ \times [0,T), \tag{4}$$

with nonlinear boundary $-|u_x|^\beta u_x(0,t) = u^p(0,t),\ t \in [0,T)$ and initial data $u(x,0) = u_0(x),\ x \in \mathbb{R}^+$, where $\beta,\ p > 0$ admits the critical exponents $p_0 = \frac{2\beta+2}{\beta+2}$ and $p_c = 2\beta + 2$, which can be obtained by taking $m = 1$ in (1). A similar work was carried in [5]. The critical exponents for the heat equation with a mixed nonlinear Dirichlet-Neumann boundary condition were considered in [6]. We refer to [7] for a survey on studies of critical exponents in particular[8-10] for the results of critical exponents to degenerate parabolic equations or systems.

In this paper we will prove that the critical exponents of (1) should be

$$p_0 = \frac{2\beta + m + 1}{\beta + 2}, \quad p_c = 2\beta + m + 1.$$

Together with such p_0 and p_c, the main results of this paper are stated by the following theorems.

Theorem 1 Any solutions of (1) are global if $0 < p \leq p_0$.

Theorem 2 Any nontrivial solutions of (1) blow up in finite time if $p_0 < p < p_c$.

Theorem 3 The solutions of (1) are global with small initial data and non-global with large initial data if $p > p_c$.

2 Proof of the main results

We prove the main results of the paper in this section.

Proof of Theorem 1 Construct

$$\overline{u}(x,t) = \mathrm{e}^{kt}\left(M + \mathrm{e}^{-Lx\mathrm{e}^{-lt}}\right)^{\frac{1}{m}} = \mathrm{e}^{kt} z^{\frac{1}{m}},$$

with

$$\begin{aligned} &M = \max\left\{\|u_0\|_\infty^m,\ 1\right\}, \qquad &&L = (M+1)^{\frac{p-\beta+m\beta}{m(\beta+1)}} m^{\frac{\beta}{\beta+1}}, \\ &k = (\beta+1)\tfrac{L^{\beta+2}}{m^\beta}, &&l = \tfrac{(\beta+m-1)k}{\beta+2}. \end{aligned} \tag{5}$$

A simple computation shows

$$\overline{u}_t \geq k\mathrm{e}^{kt} M^{\frac{1}{m}}, \quad \left(|\overline{u}_x|^\beta (\overline{u}^m)_x\right)_x \leq (\beta+1)\frac{L^{\beta+2}}{m^\beta}\mathrm{e}^{\beta(k-l)t+(mk-2l)t} M^{\beta(\frac{1}{m}-1)}, \tag{6}$$

with $m \geq 1$, and hence

$$\overline{u}_t \geq \left(|\overline{u}_x|^\beta (\overline{u}^m)_x\right)_x, \quad \text{in} \quad \mathbb{R}^+ \times (0,T) \tag{7}$$

by (5). Moreover, with $p \leq p_0$ and (5), it holds that

$$-|\overline{u}_x|^\beta (\overline{u}^m)_x(0,t) \geq \overline{u}^p(0,t), \quad \text{for} \quad t \in [0,T) \tag{8}$$

on the boundary, and

$$\overline{u}(x,0) \geq u_0(x), \quad \text{on} \quad \mathbb{R}^+ \tag{9}$$

for the initial data. We conclude from (7)-(9) that $\overline{u}$ is a global supersolution of (1) whenever $0 < p \leq p_0$.

On the contrary, we have the following lemma.

Lemma 1 The solutions of (1) blow up in finite time with large initial data if $p > p_0$.

Proof Construct $\underline{u}(x,t) = (T-t)^{-k} f(\xi)$, $\xi = x(T-t)^{-l}$, where f is the compactly supported function to be determined with

$$k = \frac{\beta+1}{(\beta+2)p - (2\beta+m+1)}, \quad l = \frac{p-\beta-m}{(\beta+2)p-(2\beta+m+1)}. \tag{10}$$

It follows from (10) that

$$k+1 = \beta(k+l) + km + 2l, \quad \beta(k+l) + km + l = pk.$$

After a direct computation, we have

$$\underline{u}_t \leq \left(|\underline{u}_x|^\beta (\underline{u}^m)_x\right)_x, \quad (x,t) \in \mathbb{R}^+ \times (0,T), \quad -|\underline{u}_x|^\beta (\underline{u}^m)_x(0,t) \leq \underline{u}^p(0,t), \quad t \in (0,T) \tag{11}$$

if $f(\xi)$ satisfies

$$kf(\xi) + l\xi f'(\xi) \leq \left(|f'|^\beta (f^m)'\right)'(\xi), \quad -\left(|f'|^\beta (f^m)'\right)(0) \leq f^p(0).$$

Set $f(\xi) = A(cA^{\beta+m-1} - \xi)_+^\alpha = Az_+^\alpha$ with

$$\alpha = \frac{\beta+1}{\beta+m-1}, \quad c = \frac{m\alpha^{\beta+2}}{k},$$

and $A > 0$ to be determined later. Noticing $\beta(\alpha-1) + m\alpha - 1 = \alpha$ and $z \leq cA^{\beta+m-1}$, we have

$$kf(\xi) + l\xi f'(\xi) - \left(|f'|^\beta (f^m)'\right)'(\xi) \leq z_+^{\alpha-1} A^{\beta+m}\left(kc - m\alpha^{\beta+2}\right) \leq 0.$$

If we choose

$$A \geq \left(m\alpha^{\beta+1} c^{(1-p)\alpha}\right)^{\frac{1}{p(\beta+2)-(2\beta+m+1)}},$$

then $-\left(|f'|^\beta (f^m)'\right)(0) - f^p(0) \leq 0$. Thus, $\underline{u}(x,t)$ is a non-global subsolution of (1) with the initial data large that $u_0(x) \geq \underline{u}(x,0)$ on $\mathbb{R}^+$.

In the sequel, we follow the techniques in [4] to prove Theorems 2 and 3.

Proof of Theorem 2 Construct

$$u_B(x,t) = (t+\tau)^{-\lambda} g(\xi), \quad g(\xi) = C(c^\gamma - \xi^\gamma)_+^L, \quad \xi = x(t+\tau)^{-\lambda} \tag{12}$$

with τ, $c > 0$ and

$$\lambda = \frac{1}{m+2\beta+1}, \quad L = \frac{\beta+1}{\beta+m-1}, \quad \gamma = \frac{\beta+2}{\beta+1}, \quad C = \left(\frac{\lambda}{m}(L\gamma)^{-\beta-1}\right)^{\frac{1}{\beta+m-1}}.$$

It is easy to verify that $g(\xi)$ satisfies

$$\lambda g(\xi) + \lambda\xi g'(\xi) - \left(|g'|^\beta (g^m)'\right)'(\xi) = 0, \quad -\left(|g'|^\beta (g^m)'\right)(0) = g^p(0). \tag{13}$$

Since $g'(0) = 0$, the self-similar solution $u_B(x,t)$ satisfies $u_B(0,t) = 0$. Without loss of generality, suppose $u(0,t_0) > 0$ for some $t_0 > 0$. Therefore, there exist large $\tau > 0$ and small $c > 0$ such that $u(x,t_0) \ge u_B(x,t_0)$ for all $x > 0$. A direct computation shows that $u_B(x,t)$ is a subsolution of (1) on $\mathbb{R}^+ \times (t_0,+\infty)$, namely, $u(x,t) \ge u_B(x,t)$, $(x,t) \in \mathbb{R}^+ \times (t_0,+\infty)$.

We claim that there exist $t^* \ge t_0$ and T large enough such that $u_B(x,t^*) \ge \underline{u}(x,0)$ on $\mathbb{R}^+$. In fact, $u_B(x,t^*) \ge \underline{u}(x,0)$ is true provided that

$$(t^*+\tau)^{-\frac{1}{2\beta+m+1}} \gg T^{-\frac{\beta+1}{(\beta+1)p-(2\beta+m+1)}}, \quad (t^*+\tau)^{-\frac{1}{2\beta+m+1}} \ll T^{-\frac{p-\beta-m}{(\beta+1)p-(2\beta+m+1)}}. \tag{14}$$

Since $p < p_c = 2\beta+m+1$ implies

$$\frac{\beta+1}{(\beta+1)p-(2\beta+m+1)} > \frac{p-\beta-m}{(\beta+1)p-(2\beta+m+1)},$$

there exist $t^* > t_0$ and T large enough that (14) are both satisfied. We conclude that $u(x,t^*) \ge u_B(x,t^*) \ge \underline{u}(x,0)$, $x \in \mathbb{R}^+$. Based on Theorem 1, it can be conducted that every nontrivial nonnegative solution of (1) blows up in finite time if $p_0 < p < p_c$.

Proof of Theorem 3 Construct $\overline{u}(x,t) = (t+\tau)^{-k}h(\xi)$, $\xi = x(t+\tau)^{-l}$, where constants $\tau > 0$, and k, l are defined as (10). If $h(\xi)$ satisfies

$$-kh - l\xi h' \ge \big(|h'|^\beta (h^m)'\big)', \quad |h'|^\beta (h^m)'(0) \ge h^p(0), \tag{15}$$

then $\overline{u}$ is a supersolution of (1) with small initial $u_0(x)$. We set

$$h(\xi) = A\big(C\big((da)^\gamma - (\xi+a)^\gamma\big)_+^L\big) = Ag(\xi+a) \tag{16}$$

with C, g, γ, L defined as (12). From (13), we know that $g(\xi+a)$ satisfies

$$\begin{aligned} &\big(|g'|^\beta (g^m)'\big)' = -\lambda g - \lambda(\xi+a)g', \quad -\big(|g'|^\beta (g^m)'\big)(a) = g^p(a), \\ &g'(\xi+a) = -CL\gamma(\xi+a)^{\gamma-1}\big((da)^\gamma - (\xi+a)^\gamma\big)_+^{L-1}. \end{aligned} \tag{17}$$

By a simple computation, we know that the first inequality of (15) is true if

$$(k - \lambda A^{\beta+m-1})g + \big((l - \lambda A^{\beta+m-1})(\xi+a) - la\big)g' \le 0. \tag{18}$$

By (16), (17), the inequality (18) is equivalent to

$$-\theta_1 A^{\beta+m-1}(\xi+a)^\gamma + a\theta_2(\xi+a)^{\gamma-1} - \theta_3(da)^\gamma \le 0,$$

with

$$\theta_1 = \frac{1 - A^{\beta+m-1}}{\beta+m-1}, \quad \theta_2 = \frac{(\beta+2)(q-\beta-m)}{(\beta+m-1)\big((\beta+2)p-(2\beta+m-1)\big)},$$

$$\theta_3 = \frac{A^{\beta+m-1}}{2\beta+m-1} - \frac{\beta+1}{(\beta+2)p-(2\beta+m-1)}.$$

Let

$$1 > A^{\beta+m-1} > \frac{(\beta+1)(2\beta+m-1)}{(\beta+2)p-(2\beta+m-1)},$$

then $\theta_i > 0\,(i=1,2,3)$. Set $\varphi(y) = -\theta_1 A^{\beta+m-1} y^{\frac{\gamma}{\gamma-1}} + a\theta_2 y - \theta_3(da)^\gamma$ with $y=(\xi+a)^{\gamma-1}$, then $\varphi(y)$ is a concave function, and attains its maximum at

$$y^* = \Big(\frac{a\theta_2(\gamma-1)}{\theta_1\gamma}\Big)^{\gamma-1}.$$

So, if we choose

$$d^\gamma \ge \frac{\theta_2}{\theta_3\gamma}\Big(\frac{\theta_2(\gamma-1)}{\theta_1\gamma}\Big)^{\gamma-1},$$

then $\varphi(y^*) \le 0$, and thus (18) is true.

Furthermore, if

$$d^\gamma > \max\Big\{1, \frac{\theta_2}{\theta_3\gamma}\Big(\frac{\theta_2(\gamma-1)}{\theta_1\gamma}\Big)^{\gamma-1}\Big\},$$

with a small enough, we have

$$(AC)^{p-\beta-m}(d^\gamma-1)^{\frac{(p-1)(\beta+1)}{\beta+m-1}} a^{\frac{(\beta+2)p-(2\beta+m+1)}{\beta+m-1}} \le m\Big(\frac{\beta+2}{\beta+m-1}\Big)^{\beta+1},$$

which yields the second inequality of (15).

In summary, $\overline{u}$ is a supersolution of (1) with u_0 small enough if $p > p_c = 2\beta+m+1$.

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一类具有非线性边界流的双重退化抛物型方程的临界指标

姜朝欣[1]， 孔令花[2]

(1- 大连理工大学数学科学学院，大连 116024； 2- 大连海洋大学理学院，大连 116023)

摘 要: 本文研究了一类具有非线性边界流的双重退化抛物型方程，该方程可用来描述多孔介质中的非牛顿渗流现象，可以描述气体或液体在多孔介质中的流动，具有广泛的实际背景．通过构造不同的自相似上、下解得到了方程的临界指标，即整体存在指标p_0和临界Fujita指标p_c．主要结果为：当$0<p\le p_0$时，方程存在整体解；当$p_0<p<p_c$时，方程不存在非负非平凡整体解；当$p>p_c$时，对于小初值方程存在整体解，对于大初值不存在整体解．

关键词: 临界指标；整体存在；双重退化抛物方程；爆破；非线性边界流